MoL–Analysis of Periodic Structures

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Abstract — **A new algorithm for the analysis of finite three-dimensional symmetrical periodic structures by the Method of Lines (MoL) is presented and substantiated. It combines the numerical stable impedance transformation with the Floquet's theorem. A numerically stable way of obtaining Floquet modes using open- or short-circuit matrix parameter description of two-ports is proposed. To validate the described method, a microstrip meander line is designed, realized and measured. Comparison between measured and simulated results is given.**

I. Introduction

Periodic structures play an important role in many microwave and optical devices. Examples are meander lines, Bragg gratings, bandgap structures, photonic crystals [1], dielectric antennas [2], magnetron resonators [3]. Meander lines (Fig. 1) are especially used for group delay equalization or as delay elements.

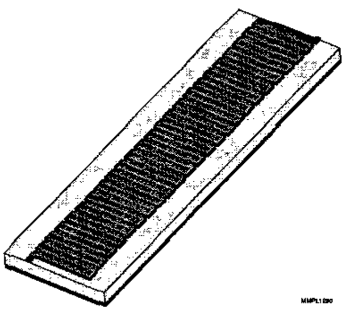

Fig. 1: Microstrip meander line.

Periodic structures can contain very high number of periods (up to several thousand in optical devices). Analyzing such structures period by period e. g. with impedance transformation from the output to the input [4] is very time consuming and requires large memory capacity. It is therefore limited to the structures with only several periods. Much better way of modeling of periodic structures is to use impedance transformation [4],[5] combined with Floquet's theorem [6]. In this case, modes of one period (Floquet modes) must be obtain and then expand into the fields at the input and the output of the structure. This makes possible to analyze finite periodic structures. A finite periodic structure can be considered as a finite homogenous waveguide and therefore concatenated with another waveguides at the begin and at the end of the structure. Such approach allows to significantly reduce computing time and memory space requirements. This concept has already been successfully used with the Method of Lines (MoL) for modeling of two-dimensional structures in optics [7]. However, at microwave frequencies, for 3–D structures with metal, the algorithm presented in [7] can not be applied. It is due to numerical problems and instability. Till now, using the MoL, only propagation constant of 3–D infinite periodic structures has been calculated [8], [9]. In this case periodic boundary conditions were used. The results presented in [8] are, in contrary to what was claimed by R. S. Chen *et al* [10], correct. The only approximation which was made was due to discretizing the structure; the stop-band was correctly obtained.

In this paper, an alternative, very efficient and accurate algorithm for modeling of symmetrical 3–D structures is proposed. It combines the numerical stable impedance transformation [4],[5] with the Floquet's theorem [6]. The Floquet modes are determined in a numerical stable way, using open- or short-circuit matrix parameter description of two ports.

Three-dimensional structures are discretized in two directions perpendicular to the direction of propagation. In the direction of propagation an analytical solution is performed. For details containing the discretization way see e. g. [11].

Fig. 2 shows an example of 2–D circular periodic structure - a magnetron resonator - and the way of discretizing of such structures.

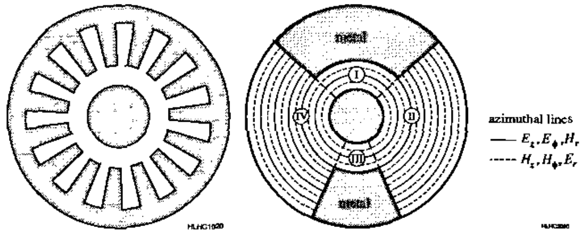

Fig. 2: Cross-section of a magnetron resonator (left) and the way of discretization general circular ridge guides (right).

For the structures with big differences between the size of individual elements of the cross-section (e. g. for meander-line), the novel nonequidistant discretization [12] can be used. It enables to significantly reduce the number of lines needed for discretization with maintenance of second order accuracy of the discretization operators.

To verify the proposed method of analysis, a microstrip meander line was designed, realized and measured. Comparison between the measured and the simulated results is given. The second analyzed structure is the magnetron resonator (Fig. 2) modeled by Raguin *et al* [3].

II. Theory

Since the general algorithm for analyzing of periodic structures can be found in [13], only the most important relations will be here presented. The main stress will be put on avoiding of numerical problems, which can arise in case of 3–D structures with metal.

To model a finite periodic structure we need a relation between the fields at the end and at the begin of one period. For a symmetrical period as shown in Fig. 3, the open-circuit impedance matrix-parameter description for a symmetrical two-port is given by

$$\begin{bmatrix}\mathbf{E}_\mathrm{A}\\ \mathbf{E}_\mathrm{B}\end{bmatrix} = \begin{bmatrix} z_1 & z_2 \\ z_2 & z_1 \end{bmatrix}\begin{bmatrix}\mathbf{H}_\mathrm{A}\\ -\mathbf{H}_\mathrm{B}\end{bmatrix} \tag{1}$$

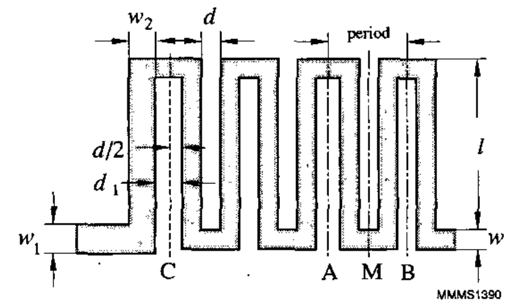

Fig. 3: First periods of the analyzed meander line.

and therefore the transfer matrix notation after transformation to Floquet-modes according to:

$$\mathbf{E}_\mathrm{A,B} = \mathbf{S}_\mathrm{E}\widetilde{\mathbf{E}}_\mathrm{A,B} \qquad \mathbf{H}_\mathrm{A,B} = \mathbf{S}_\mathrm{H}\widetilde{\mathbf{H}}_\mathrm{A,B} \tag{2}$$

reads as

$$\begin{bmatrix}\widetilde{\mathbf{E}}_\mathrm{B}\\ \widetilde{\mathbf{H}}_\mathrm{B}\end{bmatrix} = \begin{bmatrix} S_\mathrm{E}^{-1} z_1 z_2^{-1} S_\mathrm{E} & -S_\mathrm{E}^{-1}\left(z_1 z_2^{-1} z_1 - z_2\right) S_\mathrm{H} \\ -S_\mathrm{H}^{-1} z_2^{-1} S_\mathrm{E} & -S_\mathrm{H}^{-1} z_2^{-1} z_1 S_\mathrm{H}\end{bmatrix}\begin{bmatrix}\widetilde{\mathbf{E}}_\mathrm{A}\\ \widetilde{\mathbf{H}}_\mathrm{A}\end{bmatrix} \tag{3}$$

The equation of Floquet modes is defined by:

$$\begin{bmatrix}\widetilde{\mathbf{E}}_\mathrm{B}\\ \widetilde{\mathbf{H}}_\mathrm{B}\end{bmatrix} = \begin{bmatrix} \cosh \mathbf{\Gamma}_\mathrm{F} & -\widetilde{\mathbf{Z}}_0 \sinh \mathbf{\Gamma}_\mathrm{F} \\ -\widetilde{\mathbf{Y}}_0 \sinh \mathbf{\Gamma}_\mathrm{F} & \cosh \mathbf{\Gamma}_\mathrm{F}\end{bmatrix}\begin{bmatrix}\widetilde{\mathbf{E}}_\mathrm{A}\\ \widetilde{\mathbf{H}}_\mathrm{A}\end{bmatrix} \tag{4}$$

Due to equivalence of (3) and (4), the following eigenvalue problems occur:

$$\begin{aligned} z_1 z_2^{-1} S_\mathrm{E} &= S_\mathrm{E}\boldsymbol{\lambda}_\mathrm{E} \\ z_2^{-1} z_1 S_\mathrm{H} &= S_\mathrm{H}\boldsymbol{\lambda}_\mathrm{H} \end{aligned} \tag{5}$$

which results in:

$$\boldsymbol{\lambda}_\mathrm{E} = \boldsymbol{\lambda}_\mathrm{H} = \boldsymbol{\lambda} = \cosh \mathbf{\Gamma}_\mathrm{F} \tag{6}$$

Solving one of the eigenvalue problems of (5) is for 3–D structures with metal, due to numerical problems with inversion of the matrix z_2 (the matrix is badly scaled), not numerical stable. For this reason, we determine open and short circuit matrix parameters z_eh and z_oh of a half of one period. For a symmetrical period (Fig. 3) two excitation cases can be distinguished (1):

1. Even case: $\mathbf{E}_\mathrm{A} = \mathbf{E}_\mathrm{B},\ \mathbf{H}_\mathrm{A} = -\mathbf{H}_\mathrm{B} \rightarrow$ magnetic wall can be put in the symmetry plane M

$$\mathbf{E}_\mathrm{A} = (z_1 + z_2)\,\mathbf{H}_\mathrm{A} \longrightarrow \mathbf{Z}_\mathrm{even} = z_1 + z_2 = z_\mathrm{eh} \tag{7}$$

2. Odd case: $\mathbf{E}_\mathrm{A} = -\mathbf{E}_\mathrm{B},\ \mathbf{H}_\mathrm{A} = \mathbf{H}_\mathrm{B} \rightarrow$ electric wall can be put in the symmetry plane M

$$\mathbf{E}_\mathrm{A} = (z_1 - z_2)\,\mathbf{H}_\mathrm{A} \longrightarrow \mathbf{Z}_\mathrm{odd} = z_1 - z_2 = z_\mathrm{oh} \tag{8}$$

The matrices z_1 and z_2 can be then obtained as follows:

$$z_1 = 0.5\,(z_\mathrm{eh} + z_\mathrm{oh}) \qquad z_2 = 0.5\,(z_\mathrm{eh} - z_\mathrm{oh}) \tag{9}$$

Using these relations leads to

$$z_2^{-1} z_1 = (z_\mathrm{oh}^{-1} z_\mathrm{eh} - I)^{-1}(z_\mathrm{oh}^{-1} z_\mathrm{eh} + I) \tag{10}$$

The eigenvalues can be then calculated from

$$\begin{aligned} \boldsymbol{\lambda} &= S_\mathrm{H}^{-1} z_2^{-1} z_1 S_\mathrm{H} \\ &= (S_\mathrm{H}^{-1} z_\mathrm{oh}^{-1} z_\mathrm{eh} S_\mathrm{H} - I)^{-1}(S_\mathrm{H}^{-1} z_\mathrm{oh}^{-1} z_\mathrm{eh} S_\mathrm{H} + I) \end{aligned} \tag{11}$$

Which results in:

$$\boldsymbol{\lambda} = (\boldsymbol{\lambda}_\mathrm{h} + I)(\boldsymbol{\lambda}_\mathrm{h} - I)^{-1} = (\boldsymbol{\lambda}_\mathrm{h} - I)^{-1}(\boldsymbol{\lambda}_\mathrm{h} + I) \tag{12}$$

where

$$\boldsymbol{\lambda}_\mathrm{h} = S_\mathrm{H}^{-1} z_\mathrm{oh}^{-1} z_\mathrm{eh} S_\mathrm{H} \tag{13}$$

The solution of this eigenvalue problem is numerical stable. This is the fundamental equation for determination of the Floquet modes.

Because, according to the Floquet's theorem, each periodic structure can be described as a homogeneous waveguide, for a finite periodic structure with N periods, eq.(4) reads as (with $N\boldsymbol{\Gamma}_\mathrm{F} = \boldsymbol{\Gamma}_\mathrm{NF}$) :

$$\begin{bmatrix}\widetilde{\mathbf{E}}_\mathrm{D}\\ \widetilde{\mathbf{H}}_\mathrm{D}\end{bmatrix} = \begin{bmatrix} \cosh \boldsymbol{\Gamma}_\mathrm{NF} & -\widetilde{\mathbf{Z}}_0 \sinh \boldsymbol{\Gamma}_\mathrm{NF} \\ -\widetilde{\mathbf{Y}}_0 \sinh \boldsymbol{\Gamma}_\mathrm{NF} & \cosh \boldsymbol{\Gamma}_\mathrm{NF}\end{bmatrix}\begin{bmatrix}\widetilde{\mathbf{E}}_\mathrm{C}\\ \widetilde{\mathbf{H}}_\mathrm{C}\end{bmatrix} \tag{14}$$

The ports C and D are the input and output ports of the finite periodic structure. To these ports another structures or waveguides can be concatenated. To obtain scattering parameters of whole concatenated structure, the impedance must be transformed from the output to the input.

With $\widetilde{\mathbf{E}}_{\mathrm{C,D}} = \widetilde{\mathbf{Z}}_{\mathrm{C,D}}\widetilde{\mathbf{H}}_{\mathrm{C,D}}$ the impedance can be transform throughout whole periodic structure by:

$$\widetilde{\mathbf{Z}}_{\mathrm{C}} = \widetilde{\mathbf{z}}_1 - \widetilde{\mathbf{z}}_2\left(\widetilde{\mathbf{z}}_1 + \widetilde{\mathbf{Z}}_{\mathrm{D}}\right)^{-1}\widetilde{\mathbf{z}}_2 \tag{15}$$

where

$$\widetilde{\mathbf{z}}_1 = \widetilde{\mathbf{Z}}_0/\tanh \boldsymbol{\Gamma}_{\mathrm{FN}} \quad \widetilde{\mathbf{z}}_2 = \widetilde{\mathbf{Z}}_0/\sinh \boldsymbol{\Gamma}_{\mathrm{FN}} \tag{16}$$

The relations between Floquet impedances (marked by a tilde) and mode impedances (overlined) are given by

$$\widetilde{\mathbf{Z}}_{\mathrm{D}} = \boldsymbol{S}_{\mathrm{E}}^{-1}\boldsymbol{T}_{\mathrm{E}}\overline{\mathbf{Z}}_{\mathrm{D}}\boldsymbol{T}_{\mathrm{H}}^{-1}\boldsymbol{S}_{\mathrm{H}} \tag{17}$$

By using the above developed impedance transfer formula we can calculate the input impedance of a concatenation structure starting at the end. In case of the analyzed meander line, to match the structure to 50 Ω impedance line, a half of period with different dimensions was added at the begin and at the end of the periodic structure. In this case, to obtain the total input impedance, the output impedance was transformed to the port D, then using (17) and (15) transformed in Floquet's domain throughout the whole periodic structure to the port C and then again transformed to the very input of the structure.

Having the input impedance we can easily calculate the reflection coefficient. At the input of the structure we have a feeding waveguide. We assume that this waveguide has a characteristic impedance matrix $\overline{\mathbf{Z}}_0$. For the magnetic field at the input we then may write

$$\overline{\mathbf{H}}_{\mathrm{in}} = 2\left(\overline{\mathbf{Z}}_{\mathrm{in}} + \overline{\mathbf{Z}}_0\right)^{-1}\overline{\mathbf{E}}_{\mathrm{inf}} \tag{18}$$

where $\overline{\mathbf{E}}_{\mathrm{inf}}$ is the vector of the propagating modes in forward direction. If we assume that in the input section only the fundamental mode with amplitude 1 is propagating in the forward direction, then $\overline{\mathbf{E}}_{\mathrm{inf}}$ is given by

$$\overline{\mathbf{E}}_{\mathrm{inf}} = [1, 0, \ldots 0]^{\mathrm{t}} \tag{19}$$

With (18) and $\overline{\mathbf{E}}_{\mathrm{in}} = \overline{\mathbf{Z}}_{\mathrm{in}}\overline{\mathbf{H}}_{\mathrm{in}}$ we obtain

$$\overline{\mathbf{E}}_{\mathrm{in}} = 2\overline{\mathbf{Z}}_{\mathrm{in}}(\overline{\mathbf{Z}}_{\mathrm{in}} + \overline{\mathbf{Z}}_0)^{-1}\overline{\mathbf{E}}_{\mathrm{inf}} \tag{20}$$

The vector $\overline{\mathbf{E}}_{\mathrm{in}}$ contains the complex amplitude of the reflected fundamental and all higher modes. The scattering coefficient column vector $\mathbf{S}_{11}$ is now given by

$$\mathbf{S}_{11} = \overline{\mathbf{E}}_{\mathrm{C}} - \overline{\mathbf{E}}_{\mathrm{Cf}} \tag{21}$$

Repeating this calculation for the other modes in the input waveguide we can construct the generalized scattering matrix $\boldsymbol{S}_{11}$. The reflection coefficient of the fundamental mode is given in the above vector $\mathbf{S}_{11}$ as first component.

From the field at the input of the structure we can calculate the field in the whole structure using the derived impedance/admittance transfer equations and the forward and backward propagating field parts. The algorithm is stable and high accurate. From the fields at the end all other scattering parameters can be computed.

III. Results

To validate the proposed algorithm, a meander line with 23 periods has been fabricated, measured and compared with the calculated results. As a substrate RT/duroid 6010LM microwave laminate was used, with $\varepsilon_r = 10.2$ and thickness $h = 0.635$ mm. To match the input and output impedances of the meander line to the 50Ω line and to minimize insertion loss, the width of the both connecting lines w_1 and the first and last distance between the strips d_1 were changed. The dimensions of the fabricated meander line are (see Fig. 3) $w_1 = 0.6$ mm $w_2 = 0.4$ mm $w = 0.4$ mm, $d_1 = 0.6$ mm, $d = 0.4$ mm, $l = 6.4$ mm.

The structure was measured with a Hewlett Packard 8720D network analyzer. Fig. 4 shows the measured scattering parameters against the results calculated with the proposed algorithm. As seen, the measured and the calculated results are in very good agreement. Because in the analysis, a lossless structure was assumed, the transmission coefficient S_{21} can be computed from the relation:

$$S_{11}^2 + S_{21}^2 = 1 \tag{22}$$

However, to check the accuracy of the algorithm, we have calculated S_{21} transforming the fields from the input to the output. The difference between the theoretical (22) and numerical results was approximately 10^{-14}, what proofs the high accuracy of the proposed approach.

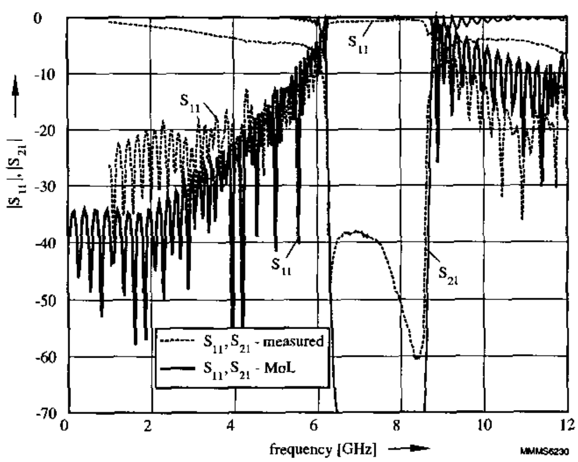

Fig. 4: Scattering parameters for the meander line.

Also the measured and calculated phase of the whole periodic structure are in very good agreement (Fig. 5).

Second analyzed structure is the magnetron resonator [3] shown in Fig. 2. This resonator is designed to operate in the fundamental $\pi/2$-mode around 38 GHz, with $N = 16$ slots of depth 1.385 mm, an anode radius of 2.25 mm and the cathode radius of 1.3 mm. Fig. 6 shows

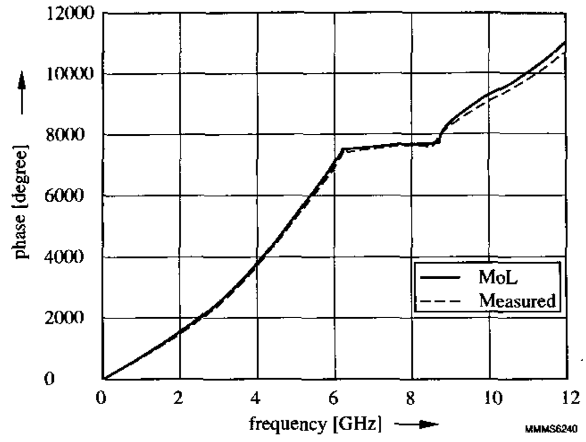

Fig. 5: Phase for the meander line.

the resonant frequencies of the TE_{l10} and TE_{l20} modes ($l = 0, 1, 2, \ldots N/2$) with axial open-boundary conditions. The results obtained by [3] and the MoL are in a very good agreement.

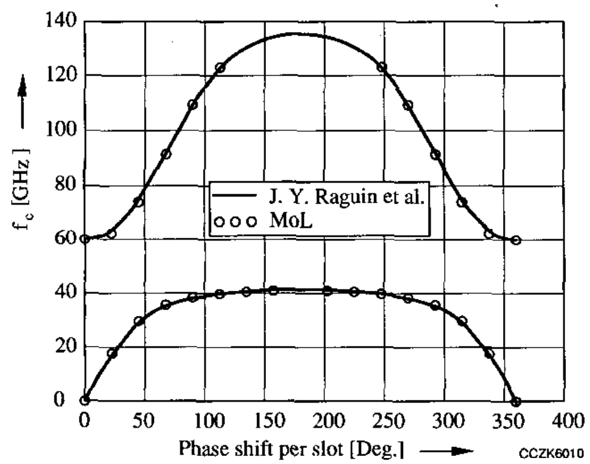

Fig. 6: Dispersion diagram associated with TE_{l10} and TE_{l20} modes.

IV. Conclusion

An efficient and numerical stable algorithm for the analysis of symmetrical periodic structures was proposed and substantiated. To analyze the finite periodic structure, Floquet modes for one period are determined and the formulas for concatenation of N periodic sections are given. An alternative, numerically stable way for obtaining Floquet modes, based on open- or short-circuit matrix parameter description is proposed.

To substantiate the proposed algorithm, two structures were analyzed. The first one, a meander line was fabricated and measured to compare calculated and measured results. The second structure is a magnetron cavity which dispersion diagram was compared with the results published in the literature.

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